ラプラス変換による微分方程式の解

定係数線形微分方程式

目的

定係数線形微分方程式を(簡便な計算で)解く

例: ある電圧波形のもとで電荷や電流がどのように変動するか。ただし、ある時刻における値、変化率が与えられている。

- $\frac{dQ}{dt} + \frac{1}{RC} Q = \frac{1}{R} V$
- $L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = \frac{dV}{dt}$

斉次では特性方程式を用いた

$$\frac{d^2 f}{dt^2} + a\frac{df}{dt} + b\,f = 0$$

$$f = e^{\lambda t} \rightarrow \lambda^2 + a\lambda + b = 0$$

$$f = A\,e^{\lambda_1 t} + B\,e^{\lambda_2 t}$$

微分方程式を特性方程式（代数方程式）に変える

$$\frac{d}{dt}\left[e^{\lambda t}\right] \rightarrow \lambda \times \left[e^{\lambda t}\right]$$

微分演算子を定数倍に変える

ラプラス変換

- 定義

$$\mathcal{L}[f] = \int_0^\infty f(t)e^{-st}dt$$

- 微分にかかわる性質

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt}e^{-st}dt, \qquad \lim_{t\to\infty} f(t)e^{-st} = 0$$

$$= [f(t)e^{-st}]_0^\infty - (-s)\int_0^\infty f(t)e^{-st}dt$$

$$= s\mathcal{L}[f] - f(0)$$

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = \mathcal{L}\left[\frac{df'}{dt}\right] = s\mathcal{L}[f'] - f'(0)$$

$$= s^2\mathcal{L}[f] - sf(0) - f'(0)$$

微分方程式を代数方程式にする

$$\frac{d^2 f(t)}{dt^2} + a\frac{df(t)}{dt} + b\,f(t) = g(t)$$

$$F(s) \equiv \mathcal{L}[f], \qquad G(s) = \mathcal{L}[g]$$

$$\{s^2F(s) - sf(0) - f'(0)\} + a\{sF(s) - f(0)\} + bF(s) = G(s)$$

$$(s^2{+}as + b)F = G(s) + (s + a)f(0) + f'(0)$$

$$F(s) = \frac{G(s) + (s + a)f(0) + f'(0)}{(s^2{+}as + b)}$$

逆変換

$$F(s) = \mathcal{L}[f], \qquad f = \mathcal{L}^{-1}[F], \qquad \mathcal{L}^{-1}\big[\mathcal{L}[f]\big] = f$$

$$f(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\, ds\,, \qquad \gamma > 0$$

ラプラス変換の例題（逆変換の準備）

変換表	原関数 $f(t) = \mathcal{L}^{-1}\{F(s)\}$ 時間領域	像関数 $F(s) = \mathcal{L}\{f(t)\}$'s'−領域/周波数領域	収束域
単位インパルス	$\delta(t)$	1	all s
単位ステップ関数	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
ランプ関数	$t \cdot u(t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\} > 0$
n 乗 （ n は整数 ）	$\frac{t^n}{n!} \cdot u(t)$	$\frac{1}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$ $(n > -1)$
q 乗 （ q は複素数 ）	$\frac{t^q}{\Gamma(q+1)} \cdot u(t)$	$\frac{1}{s^{q+1}}$	$\mathrm{Re}\{s\} > 0$ $(\mathrm{Re}\{q\} > -1)$
n 乗根	$\sqrt[n]{t} \cdot u(t) = t^{1/n} \cdot u(t)$	$\frac{1}{s^{1+1/n}} \cdot \Gamma\left(1 + \frac{1}{n}\right)$	$\mathrm{Re}\{s\} > 0$
指数減衰	$e^{-\alpha t} \cdot u(t)$	$\frac{1}{s+\alpha}$	$\mathrm{Re}\{s\} > -\alpha$
n 乗の指数減衰	$\frac{t^n}{n!} e^{-\alpha t} \cdot u(t)$	$\frac{1}{(s+\alpha)^{n+1}}$	$\mathrm{Re}\{s\} > -\alpha$
理想遅延	$\delta(t-\tau)$	$e^{-\tau s}$	
遅延付き単位ステップ関数	$u(t-\tau)$	$\frac{1}{s} \cdot e^{-\tau s}$	$\mathrm{Re}\{s\} > 0$
遅延付き n 乗の指数減衰	$\frac{(t-\tau)^n}{n!} e^{-\alpha(t-\tau)} \cdot u(t-\tau)$	$\frac{1}{(s+\alpha)^{n+1}} \cdot e^{-\tau s}$	$\mathrm{Re}\{s\} > -\alpha$
指数関数的接近	$(1 - e^{-\alpha t}) \cdot u(t)$	$\frac{\alpha}{s(s+\alpha)}$	$\mathrm{Re}\{s\} > 0$
正弦	$\sin(\omega t) \cdot u(t)$	$\frac{\omega}{s^2+\omega^2}$	$\mathrm{Re}\{s\} > 0$
余弦	$\cos(\omega t) \cdot u(t)$	$\frac{s}{s^2+\omega^2}$	$\mathrm{Re}\{s\} > 0$
双曲線正弦関数 (ハイパボリックサイン)	$\sinh(\alpha t) \cdot u(t)$	$\frac{\alpha}{s^2-\alpha^2}$	$\mathrm{Re}\{s\} > \lvert\alpha\rvert$
双曲線余弦関数 (ハイパボリックコサイン)	$\cosh(\alpha t) \cdot u(t)$	$\frac{s}{s^2-\alpha^2}$	$\mathrm{Re}\{s\} > \lvert\alpha\rvert$
正弦波の指数減衰	$e^{-\alpha t} \sin(\omega t) \cdot u(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$	$\mathrm{Re}\{s\} > -\alpha$
余弦波の指数減衰	$e^{-\alpha t} \cos(\omega t) \cdot u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$	$\mathrm{Re}\{s\} > -\alpha$
自然対数	$\ln\left(\frac{t}{t_0}\right) \cdot u(t)$	$-\frac{1}{s}\left[\,\ln(t_0 s) + \gamma\,\right]$	$\mathrm{Re}\{s\} > 0$

横軸の縮小･拡大　$\mathcal{L}[at]$

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

$$\begin{aligned}
\mathcal{L}[f(at)] &= \int_0^{\infty} f(at) e^{-st} dt \\
&= \int_0^{\infty} f(at) e^{-\frac{s}{a}(at)} \frac{d(at)}{a} \\
&= \frac{1}{a} \int_0^{\infty} f(t') e^{-\frac{s}{a}t'} dt' = \frac{1}{a} F\left(\frac{s}{a}\right)
\end{aligned}$$

s軸のシフト

$$\mathcal{L}[f(t)e^{at}] = \int_0^\infty f(t)e^{(a-s)t}dt$$

$$= \int_0^\infty f(t)e^{-(s-a)t}dt = F(s-a)$$

積分のラプラス変換

$$\mathcal{L}\left[\int_0^t f(t')\,dt'\right] = \int_0^\infty \left(\int_0^t f(t')\,dt'\right) e^{-st} dt$$

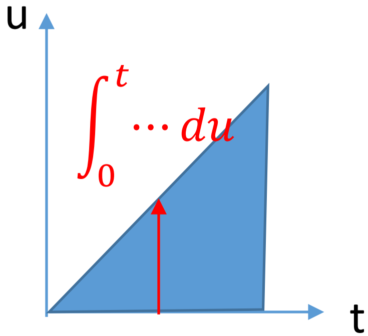
$$= \int_0^\infty \left(\int_0^t f(t')\,dt'\right) \left\{\left(-\frac{1}{s}\right)\frac{d}{dt}(e^{-st})\right\} dt$$

$$= \left[\left(\int_0^t f(t')\,dt'\right)\left(-\frac{e^{-st}}{s}\right)\right]_0^\infty - \int_0^\infty f(t)\left(-\frac{e^{-st}}{s}\right) dt$$

$$= \frac{1}{s}\int_0^\infty f(t)\, e^{-st} dt = \frac{1}{s} F(s)$$

畳み込み積分のラプラス変換

$$f * g \equiv \int_0^t f(t-u)g(u)du$$

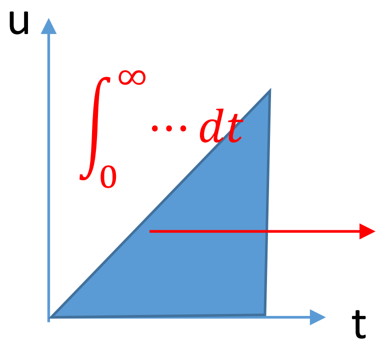


$$\mathcal{L}[f * g] = \int_0^\infty \left(\int_0^t f(t-u)g(u)du\right) e^{-st}dt$$

$$= \int_0^\infty \left(\int_u^\infty f\underbrace{(t-u)}_{w} e^{-st}dt\right)g(u)du$$

$$= \int_0^\infty \left(\int_0^\infty f(w)e^{-sw}dw\right)e^{-su}g(u)du$$

$$= \int_0^\infty f(w)e^{-sw}dw \times \int_0^\infty g(u)e^{-su}dw = F(s)G(s)$$



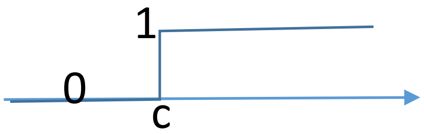
例題 : $\mathcal{L}[1], \mathcal{L}[a]$

$$\int_0^{\infty} a e^{-st} dt = -\frac{a}{s}[e^{-st}]_{t=0}^{\infty} = \frac{a}{s}$$

$$\mathcal{L}[a] = \frac{a}{s}$$

$$\mathcal{L}^{-1}\left[\frac{a}{s}\right] = a$$

例題:$\mathcal{L}[u(t-c)]$, 階段関数



$$\int_0^{\infty} u(t-c)e^{-st}dt = \int_c^{\infty} e^{-st}dt = -\frac{1}{s}[e^{-st}]_{t=c}^{\infty} = \frac{e^{-cs}}{s}$$

$$\mathcal{L}[u(t-c)] = \frac{e^{-cs}}{s} \cdots c > 0$$

$$\mathcal{L}^{-1}\left[\frac{e^{-cs}}{s}\right] = u(t-c)$$

例題：$\mathcal{L}[\delta(t-c)]$, デルタ関数

$$\delta(t-c)\begin{cases}\neq 0 \cdots t=c \\ = 0 \cdots t \neq 0\end{cases}, \qquad \int_{-\infty}^{\infty} \delta(t-c)\, dt = 1$$

$$\mathcal{L}[\delta(t-c)] = \int_{0}^{\infty} \delta(t-c)\, e^{-st} dt = e^{-cs}$$

$$\mathcal{L}[\delta(t)] = \lim_{c \to +0} e^{-cs} = 1$$

例題：$\mathcal{L}[t]$

$$\int t\,e^{-st}dt = t \times \frac{e^{-st}}{-s} - \int 1 \times \frac{e^{-st}}{-s}dt = \frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}$$
$$= \frac{e^{-st}}{s^2}(-st-1)$$

$$\int_0^{\infty} te^{-st}dt = \left[\frac{e^{-st}}{s^2}(-st-1)\right]_{t=0}^{\infty} = \frac{1}{s^2}$$

$$\mathcal{L}[t] = \frac{1}{s^2}$$
$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$
$$\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{1}{n!}t^n$$

例題 : $\mathcal{L}[e^{-at}]$

$$\int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{(-a-s)t}dt$$

$$= \frac{-1}{a+s}\left[e^{-(a+s)t}\right]_{t=0}^\infty = \frac{1}{s+a} \cdots (s > -a)$$

$$\mathcal{L}[e^{-at}] = \frac{1}{s+a}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$$

$$\mathcal{L}[te^{-at}] = \frac{1}{(s+a)^2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^2}\right] = te^{-at}$$

例題: $\mathcal{L}[\sin \omega t] = \frac{1}{2i}\left(\mathcal{L}\left[e^{i\omega t}\right] - \mathcal{L}\left[e^{-i\omega t}\right]\right)$

$$\int_0^{\infty} e^{i\omega t} e^{-st} dt = \int_0^{\infty} e^{(i\omega - s)t} dt$$

$$= \frac{1}{i\omega - s}\left[e^{i\omega t} e^{-st}\right]_{t=0}^{\infty} = \frac{1}{s - i\omega}$$

$$\mathcal{L}[\sin \omega t] = \frac{1}{2i}\left(\frac{1}{s - i\omega} - \frac{1}{s + i\omega}\right) = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}^{-1}\left[\frac{\omega}{s^2 + \omega^2}\right] = \sin \omega t$$

$$\mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2}$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2 + \omega^2}\right] = \cos \omega t$$

例題:

$$\mathcal{L}[e^{-at}\sin\omega t] = \frac{1}{2i}\left(\mathcal{L}\left[e^{-at}e^{i\omega t}\right] - \mathcal{L}\left[e^{-at}e^{-i\omega t}\right]\right)$$

$$\int_0^{\infty} e^{(-a+i\omega)t}e^{-st}dt = \int_0^{\infty} e^{(i\omega-a-s)t}dt$$

$$= \frac{1}{i\omega - a - s}\left[e^{i\omega t}e^{-(a+s)t}\right]_{t=0}^{\infty} = \frac{1}{s+a-i\omega} \cdots s+a>0$$

$$\mathcal{L}[e^{-at}\sin\omega t] = \frac{1}{2i}\left(\frac{1}{s+a-i\omega} - \frac{1}{s+a+i\omega}\right)$$

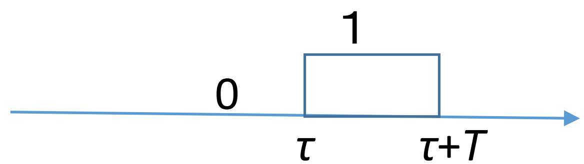
$$= \frac{\omega}{(s+a)^2+\omega^2}$$

$$\mathcal{L}^{-1}\left[\frac{\omega}{(s+a)^2+\omega^2}\right] = e^{-at}\sin\omega t$$

$$\mathcal{L}[e^{-at}\cos\omega t] = \frac{s+a}{(s+a)^2+\omega^2}$$

$$\mathcal{L}^{-1}\left[\frac{s+a}{(s+a)^2+\omega^2}\right] = \mathrm{e}^{\mathrm{at}}\cos\omega t$$

例題：
$[\tau, \tau+T]$、高さ1のパルス $u_{\tau,T}(t)$



$$\mathcal{L}\left[u_{\tau,T}\right] = \int_0^{\infty} u_{\tau,T}(t)\, e^{-st} dt = \int_{\tau}^{\tau+T} 1 \cdot e^{-st} dt$$

$$= \frac{1}{-s}\left[e^{-st}\right]_{t=\tau}^{\tau+T} = \frac{1-e^{-Ts}}{s} e^{-s\tau}$$

$$\mathcal{L}\left[u_{\tau,T}\right] = \frac{1-e^{-Ts}}{s} e^{-s\tau}$$

例題:周期関数

$$f(t) = f(t+T) = f(t+2T) = \cdots$$

$$\mathcal{L}\,[f] = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{T} f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \cdots$$

$$= \int_0^{T} f(t)e^{-st}dt + \int_0^{T} f(t'+T)e^{-s(t'+T)}d\mathrm{t}' + \cdots$$

$$= \sum_{n=0,\infty} e^{-nsT} \times \int_0^{T} f(t')e^{-st'}dt' = \frac{1}{1-e^{-sT}} \int_0^{T} f(t')e^{-st'}dt'$$

$$\because \int_0^{T} f(t'+nT)e^{-s(t'+nT)}dt' = e^{-nTs} \int_0^{T} f(t')e^{-st'}dt'$$

微分方程式を解く

例題

$$y'' - 4y' + 3y = e^{2t}, \qquad y(0) = 0, y'(0) = 0$$

$Y \equiv \mathcal{L}[y]$:

$$\{s^2Y - sy(0) - y'(0)\} - 4\{sY - y(0)\} + 3Y = \frac{1}{s-2}$$

$$(s^2 - 4s + 3)Y = \frac{1}{s-2}$$

$$\to Y = \frac{1}{(s-1)(s-2)(s-3)} = \frac{\frac{1}{2}}{s-1} + \frac{-1}{s-2} + \frac{\frac{1}{2}}{s-3}$$

$$y = \mathcal{L}^{-1}[Y] = \frac{1}{2}e^{t} - e^{2t} + \frac{1}{2}e^{3t}$$

部分分数分解

$$\frac{1}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$\left[\frac{(s-1)}{(s-1)(s-2)(s-3)}\right]_{s=1} = A + \left[\left(\frac{B}{s-2} + \frac{C}{s-3}\right)(s-1)\right]_{s=1}$$

$$\left[\frac{1}{(s-2)(s-3)}\right]_{s=1} = \frac{1}{2} = A$$

部分分数分解
(分母にべき乗の項があるとき)

$$F(x) = \frac{1}{(x-2)^2(x-1)} = \frac{A}{(x-2)^2} + \frac{B}{(x-2)} + \frac{C}{(x-1)}$$

$$(x-1)\,F(x) = \frac{1}{(x-2)^2} = \frac{A(x-1)}{(x-2)^2} + \frac{B(x-1)}{(x-2)} + C$$

$$C = \lim_{x\to 1}(x-1)F(x) = \lim_{x\to 1}\frac{1}{(x-2)^2} = 1$$

$$(x-2)^2F(x) = A + (x-2)B + \frac{C(x-2)^2}{(x-1)}$$

$$A = \lim_{x\to 2}(x-2)^2F(x) = \lim_{x\to 2}\frac{1}{(x-1)} = 1$$

$$\frac{d}{dx}(x-2)^2F(x) = \frac{d}{dx}A + \frac{d}{dx}(x-2)B + \frac{d}{dx}\frac{C(x-2)^2}{(x-1)} = B + \frac{d}{dx}\frac{C(x-2)^2}{(x-1)}$$

$$\frac{d}{dx}\frac{1}{(x-1)} = \frac{-1}{(x-1)^2}\quad,\qquad B = \lim_{x\to 2}\frac{d}{dx}(x-2)^2F(x) = -1$$