4 章　演習問題　[1]　 次の不定積分を求めよ.

(1) $\int (x^2-3x+1)dx$

$$\int (x^2-3x+1)\,dx=\frac{1}{3}x^3-\frac{3}{2}x^2+x+C$$

(2) $\int (2x+5)^2\,dx$

置換：$t=(2x+5)\quad\rightarrow\quad dt=2\,dx$

$$\int (2x+5)^2\ dx=\frac{1}{2}\int t^2dt=\frac{1}{6}t^3+C'=\boxed{\frac{1}{6}(2x+5)^3+C''}=\frac{1}{6}(8x^3+3\cdot 4x^2\cdot 5+3\cdot(2x)\cdot 5^2+125)+C''$$

$$=\boxed{\frac{4}{3}x^3+10x^2+25x+\frac{125}{6}+C''}=\boxed{\frac{4}{3}x^3+10x^2+25x+C}$$

注：試験問題の答案としては，□内のいずれでも OK.

(3) $\int \sqrt{x^2-2x^4}dx$

$I=\int \sqrt{x^2-2x^4}\,dx=\int x\sqrt{1-2x^2}dx$　と変形し，$t=1-2x^2$と置換．$dt=-4x\,dx$

$$I=-\frac{1}{4}\int \sqrt{t}\,dt=-\frac{1}{4}\int t^{\frac{1}{2}}dt=-\frac{1}{4}\times\frac{t^{1+\frac{1}{2}}}{1+\frac{1}{2}}+C=-\frac{1}{6}\left(\sqrt{t}\right)^3+C=-\frac{1}{6}t\sqrt{t}+C=-\frac{1}{6}(1-2x^2)\sqrt{1-2x^2}+\mathrm{C}$$

$$=\frac{1}{6}(2x^2-1)\sqrt{1-2x^2}+\mathrm{C}$$

(4) $\int \frac{x^3-x+1}{x^2+1}dx$

$$\frac{x^3-x+1}{x^2+1}=\frac{x(x^2+1)-2x+1}{x^2+1}=x-\frac{2x}{x^2+1}+\frac{1}{x^2+1}$$

$$\int x\,dx=\frac{1}{2}x^2+C_1,\qquad \int\frac{2x}{x^2+1}dx=\int\frac{(x^2+1)'}{(x^2+1)}dx=\log(x^2+1)+C_2,\qquad \int\frac{dx}{x^2+1}=\arctan x+C_3$$

$$\int\frac{x^3-x+1}{x^2+1}dx=\frac{1}{2}x^2-\log(x^2+1)+\arctan x+C$$

(5) $\int \frac{dx}{x\sqrt{1-x^2}}$

置換:$\mathrm{t}=\sqrt{1-x^2}\Rightarrow dt=\frac{-x}{\sqrt{1-x^2}}dx=-\frac{x}{t}\mathrm{d}x\Rightarrow \int\frac{dx}{x\sqrt{1-x^2}}=\int\frac{1}{xt}\times\frac{-t}{x}\mathrm{dt}=\int\frac{1}{-x^2}dt=\int\frac{dt}{t^2-1}$

$$=\frac{1}{2}\int\left\{\frac{1}{t+1}-\frac{1}{t-1}\right\}dt=\frac{1}{2}\log\left|\frac{t+1}{t-1}\right|+C=\frac{1}{2}\log\left|\frac{\sqrt{1-x^2}+1}{\sqrt{1-x^2}-1}\right|+\mathrm{C}=\frac{1}{2}\log\left|\frac{1-\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right|+\mathrm{C}$$

別解

置換：$x=\sin\theta$，$dx=\cos\theta\,d\theta\qquad\Rightarrow\qquad \frac{dx}{x\sqrt{1-x^2}}=\frac{\cos\theta\,d\theta}{\sin\theta\cos\theta}=\frac{d\theta}{\sin\theta}\qquad\Rightarrow\int\frac{dx}{x\sqrt{1-x^2}}=\int\frac{d\theta}{\sin\theta}$

置換：$t=\tan\frac{\theta}{2}$，$dt=\frac{d\theta}{2\cos^2\frac{\theta}{2}}=\frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)d\theta=\frac{(1+t^2)}{2}d\theta,$

$$\frac{1}{\sin\theta}=\frac{1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=\frac{1}{2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2}}=\frac{1+\tan^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}}=\frac{1+t^2}{2t}\qquad\Rightarrow\qquad\frac{d\theta}{\sin\theta}=\frac{1+t^2}{2t}\frac{2dt}{1+t^2}=\frac{dt}{t}$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = \int \frac{d\theta}{\sin\theta} = \int \frac{dt}{t} = \log t, \quad x = \sin\theta = \frac{2t}{t^2+1} \quad \Rightarrow \frac{2}{x} = t + \frac{1}{t} \quad \Rightarrow t^2 - \frac{2}{x}t + 1 = 0$$

$t = \frac{1}{x} \pm \sqrt{\left(\frac{1}{x}\right)^2 - 1}, \quad x = \frac{2t}{t^2+1} \Rightarrow$ 「$t \to 0$ のとき$x \to 0$」を考慮して$t = \frac{1}{x} - \sqrt{\left(\frac{1}{x}\right)^2 - 1}$（複合の + なら $t \to \infty$）

$$t = \frac{1}{x} - \sqrt{\left(\frac{1}{x}\right)^2 - 1} = \frac{1-\sqrt{1-x^2}}{x}$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = \log t + C = \log\left|\frac{1-\sqrt{1-x^2}}{x}\right| + C$$

巻末解答と異なるように見えるが(巻末解答は，分母の√の中の負号が誤り)

$\frac{1}{2}\log\left|\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right| = \log\sqrt{\left|\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right|}$ と $\log\left|\frac{1-\sqrt{1-x^2}}{x}\right|$ が同じになる．実際

$$\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} = \frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \times \frac{1-\sqrt{1-x^2}}{1-\sqrt{1-x^2}} = \frac{\left(1-\sqrt{1-x^2}\right)^2}{1-(1-x^2)} = \left(\frac{1-\sqrt{1-x^2}}{x}\right)^2$$

また

$$\frac{1-\sqrt{1-x^2}}{x} = \frac{1-\sqrt{1-x^2}}{x} \times \frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}} = \frac{1-(1-x^2)}{x(1+\sqrt{1-x^2})} = \frac{x}{1+\sqrt{1-x^2}}$$

となるので，以下の式はすべて同じ：

$$\int \frac{dx}{x\sqrt{1-x^2}} = \log\left|\frac{1-\sqrt{1-x^2}}{x}\right| + C = \log\left|\frac{x}{1+\sqrt{1-x^2}}\right| + C = \frac{1}{2}\log\left|\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right| + C$$

(6) $\int \frac{\boldsymbol{dx}}{\boldsymbol{\tan x}}$

$$\int \frac{1}{\tan x}dx = \int \frac{\cos x}{\sin x}dx = \int \frac{(\sin x)'}{\sin x}dx = \log|\sin x| + C$$

(7) $\int x^2 \sin x\, dx$

$$u = x^2, \quad v' = \sin x \quad \Rightarrow \quad u' = 2x, \quad v = \int \sin x\, dx = -\cos x$$

$$\Rightarrow \int x^2 \sin x\, dx = x^2(-\cos x) - \int (2x)(-\cos x)dx = -x^2\cos x + 2\int x\cos x\, dx$$

$$u = x, v' = \cos x \Rightarrow u' = 1, \ v = \int \cos x\, dx = \sin x \Rightarrow \int x\cos x\, dx = x\sin x - \int \sin x\, dx = x\sin x + \cos x$$

$$\int x^2 \sin x\, dx = -x^2\cos x + 2(x\sin x + \cos x) + C = -x^2\cos x + 2x\sin x + 2\cos x + C$$

(8) $\int e^{2x}x^3 dx$

$$u = x^3, v' = e^{2x} \Rightarrow u' = 3x^2, v = \frac{1}{2}e^{2x} \Rightarrow \int e^{2x}x^3 dx = \frac{x^3}{2}e^{2x} - \int \frac{3}{2}x^2 e^{2x}dx$$

$$u = x^2, v' = e^{2x} \Rightarrow u' = 2x, v = \frac{1}{2}e^{2x} \Rightarrow \int x^2 e^{2x}dx = \frac{x^2}{2}e^{2x} - \int \frac{2}{2}xe^{2x}dx$$

$$u = x, v' = e^{2x} \Rightarrow u' = 1, v = \frac{1}{2}e^{2x} \Rightarrow \int x\,e^{2x}dx = \frac{x}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C'$$

$$\int \mathrm{e}^{2x}x^3dx = \frac{x^3}{2}e^{2x} - \frac{3}{2}\left(\frac{x^2}{2}e^{2x} - \left(\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C'\right)\right) = \frac{x^3}{2}e^{2x} - \frac{3}{2}\frac{x^2}{2}e^{2x} + \frac{3}{2}\left(\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C'\right)$$

$$= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C = \frac{1}{8}e^{2x}(4x^3 - 6x^2 + 6x - 3) + C$$

(9) $\int \cos^2 x\,dx$

$$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1 \qquad \Rightarrow \cos^2 x = \frac{1}{2}(\cos 2x + 1)$$

$$\int \cos^2 x\,dx = \frac{1}{2}\int (\cos 2x + 1)\,dx = \frac{1}{4}\sin 2x + \frac{1}{2}x + \mathrm{C}$$

(10) $\int \sqrt{a^2 - x^2}dx$

置換：$x = a\sin\theta$， $dx = a\cos\theta\,d\theta$， $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\sqrt{1 - \sin^2\theta} = a\cos\theta$

$$\int \sqrt{a^2 - x^2}\,dx = \int a\cos\theta\;\; a\cos\theta\,d\theta = a^2\int \cos^2\theta\,d\theta = a^2\left(\frac{1}{4}\sin 2\theta + \frac{1}{2}\theta\right) + \mathrm{C}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1 - \sin^2\theta} = 2\left(\frac{x}{a}\right)\sqrt{1 - \left(\frac{x}{a}\right)^2} = \frac{2}{a^2}x\sqrt{a^2 - x^2}, \qquad \theta = \arcsin\left(\frac{x}{a}\right)$$

$$\int \sqrt{a^2 - x^2}\,dx = \frac{a^2}{4}\sin 2\theta + \frac{a^2}{2}\theta + \mathrm{C} = \frac{a^2}{4}\frac{2}{a^2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \mathrm{C} = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \mathrm{C}$$