1.

（１）被積分関数の対称性と積分領域の対称性から、２次元極座標を選ぶ。

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta = \cos\theta dr - r\sin\theta d\theta$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta = \sin\theta dr + r\cos\theta d\theta$$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$dxdy = Jdrd\theta = rdrd\theta$$

$$\sqrt{x^2 + y^2} = r$$

$$\int_R \sqrt{x^2 + y^2}\,dxdy = \int_R r\,rdrd\theta = \int_{\theta=0}^{2\pi}\left(\int_{r=2}^{4} r^2 dr\right)d\theta = 2\pi\left[\frac{1}{3}r^3\right]_2^4 = \frac{2\pi}{3}(64-8) = \frac{112}{3}\pi$$

（２）３次元極座標

$$x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta$$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \varphi}d\varphi = \sin\theta\cos\varphi\,dr + r\cos\theta\cos\varphi\;d\theta - r\sin\theta\sin\varphi\,d\varphi$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial \varphi}d\varphi = \sin\theta\sin\varphi\,dr + r\cos\theta\sin\varphi\;d\theta + r\sin\theta\cos\varphi\,d\varphi$$

$$dz = \frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial \theta}d\theta + \frac{\partial z}{\partial \varphi}d\varphi = \cos\theta\,dr - r\sin\theta\,d\theta$$

$$J = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \left\{0 + (r\cos\theta\cos\varphi)(r\sin\theta\cos\varphi)(\cos\theta) + (-r\sin\theta\sin\varphi)(\sin\theta\sin\varphi)(-r\sin\theta)\right\}$$

$$-\left\{(-r\sin\theta\sin\varphi)(r\cos\theta\sin\varphi)(\cos\theta) + 0 + (\sin\theta\cos\varphi)(r\sin\theta\cos\varphi)(-r\sin\theta)\right\}$$

$$= \left\{r^2\sin\theta\cos^2\theta\cos^2\varphi + r^2\sin^3\theta\sin^2\varphi\right\} + \left\{r^2\sin\theta\cos^2\theta\sin^2\varphi + r^2\sin^3\theta\cos^2\varphi\right\}$$

$$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta$$

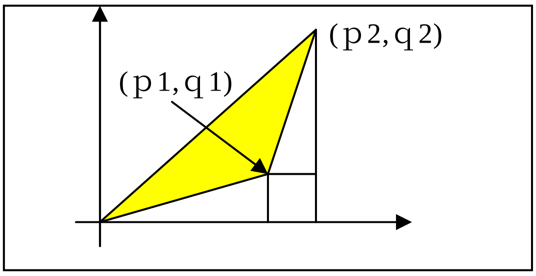
$$= r^2\sin\theta$$

$$dxdydz = Jdrd\theta\varphi = r^2 \sin\theta drd\theta d\varphi$$

$$r = \left(x^2+y^2+z^2\right)^{1/2}, \quad \left(x^2+y^2+z^2\right)^{3/2} = r^3$$

$$\int_R \frac{dxdydz}{\left(x^2+y^2+z^2\right)^{3/2}} = \int_R \frac{r^2 \sin\theta drd\theta d\varphi}{r^3} = \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\varphi \int_a^b \frac{dr}{r} = 2\cdot 2\pi \cdot \left[\ln r\right]_a^b = 4\pi \ln\frac{b}{a}$$

２．まず右の図の三角形の面積を求めよう。



一番外側の直角三角形の面積は

$p_2 q_2 /2$

である。これから２つの三角形の面積

$p_1 q_1 /2 \quad + \quad (p_2 - p_1)(q_2 - q_1)/2$

および四角形の面積

$q_1 (p_2 - p_1)$

を引くと

$(1/2)\ [p_2 q_2 - p_1 q_1 - (p_2 - p_1)(q_2 - q_1) - 2 q_1 (p_2 - p_1)] = (1/2)\ [p_1 q_2 - p_2 q_1]$

となる。

原点を移動して(x1,y1)としても、上の計算の p,q が各点の座標の差であると思えばよい。

平行四辺形の面積は２倍して得られる。

３．１で示したとおり。