Mesh quality improvement
General introduction

- **Mesh quality.** Mesh is constructed with a precise purpose: solve a given problem. The mesh quality is good if the resulting solution quality is good. Shape of elements has a strong influence on a discretization error (the relative stiffness of neighboring elements) in FEM applications.

- **Lecture is based on the following publications**

**Bibliography:**
Mesh quality improvement

In this lecture we consider:

- Improving the quality of surface and volume meshes:
  - The term smoothing is used in the sense of geometric improving (mesh node relocation) the quality of a mesh (shape of elements) to acceptable level.
  - The problem of improving meshes is formulated as follows:
  - **Input**: Unstructured mesh (data set $S(\mathbf{v})$, where $\mathbf{v}$ is a finite vertex set) with elements (cells) generated by some existing algorithm.
  - **Output**: Unstructured mesh $S'$ with higher-quality elements and rather homogeneous distribution of the mesh quality parameter values.
Problem statement

Mesh quality improvement (Cont.)

- A better node location
- Space-mapping interpolation technique
  - Limitation: Initial mesh with skinny elements
  - Multi step iterative
  - Laplacian smoothing
    - Averaging of location vertex neighbors
    - Produce an invalid mesh, shrinkage
  - Optimization-based
    - Statistic
    - Different objective functions, time consuming
    - Improving one element reduces the quality of a neighbor

Mesh improvement

- Geometric improvement
  - Mesh node relocation

- Topological improvement
  - edge collapsing
  - edge swapping
  - face swapping
  - edge splitting
Preliminaries

• Mesh modification (coarsening and mesh point movement) is an almost obligatory step for obtaining a valid finite element mesh
• “forty-odd years after the invention of the finite element method, our understanding of the relationship between mesh geometry, numerical accuracy, and stiffness matrix conditioning remains incomplete, even in the simplest cases.” [ Armstrong ]

Preliminaries

• The problem remains a difficult and computationally expensive problem [ž]
• Mostly, improvement methods for optimal point placement are based on the idea of local optimization and require an improvement of such mesh quality parameters as aspect ratio, area, angle, etc.

Preliminaries

• The problem is almost closed for planar and isotropic triangle meshes.
• However, for the mesh shown in Figure, where almost half of triangles has very “bad” or skinny triangles,
  - Laplacian smoothing and smart Laplacian smoothing produce 211 inverted triangles
  - weighted angle-based approach - 1449
Preliminaries

• Global mesh optimization has also been studied recently [1]
• In spite of the well-known fact that local enhancement does not often provide reliable results, local methods are preferable in many applications with large set of polygons
• In fact, these methods can be called deterministic

Preliminaries

• It is very difficult or even impossible to generate a mesh with elements corresponding exactly to ideal values (in the case of a triangular mesh, equal to unity values)

• Automatic generation of hexahedral meshes sometimes produces meshes with badly shaped and inverted elements. Hexahedra are highly flexible and can be distorted; there are also difficulties in generating hexahedral meshes for complex configurations [1, 2]


Preliminaries

• A novel simple method for improving (modification) triangular, quadrilateral surface, and hexahedral solid meshes is discussed

• The method for improving meshes by producing elements with a Gaussian-like (normal) distribution of the mesh quality parameter values is based on an implementation of quasi-statistical modeling
Preliminaries

• The main intention is to attain a fairly smooth change from one mesh element to another
• We assume that after modification the distribution of mesh quality parameter values varies from a rather random distribution to a smoother one
Preliminaries

• It is worth to mention that we are not using the term “mesh optimization” because optimization means the process of making something as effective as possible
• We suppose that proposed approach can be used as a software component in some mesh optimization tools
• Basically, only point movement is considered with no topology change
The problem of improving triangular and quadrilateral surface meshes can be formulated as follows:

**Input:** Unstructured mesh (data set $S(\mathbf{v})$, where $\mathbf{v}$ is a finite vertex set) with elements (cells) generated by some existing algorithm

**Output:** Unstructured mesh $S'$ with higher-quality elements and uniform distribution of the mesh quality parameter values
Triangular and Quadrilateral Improvement Algorithm (TQIA)

- A real number (a value of the mesh quality parameter) assigned to a mesh element is considered as a pseudo-random number, by analogy with the value derived from some given formula.
- Such experimental data are grouped into special intervals showing the number of elements that lie in intervals of probabilistic space fragmentation.
- Since the empirical distribution of experimental data is discrete, we consider the frequency histogram as an analogue of the density function. All classical statistical parameters for discrete cases of sampling are defined in a common way.
Triangular and Quadrilateral Improvement Algorithm

- Illustration of the algorithm
Triangular and Quadrilateral Improvement Algorithm

Modeling Formula

Equation from [C]

\[ \rho = \sqrt{-2 \ln(1 - \gamma)} \]

is used to produce values of a Gaussian random variable from the given pseudo-random or random \( \gamma \)-numbers

Triangular and Quadrilateral Improvement Algorithm

- **Deterministic Numbers**
  - In most applications, AR is used as the mesh quality parameter
  - For triangular mesh elements
    - AR is defined as the ratio of the maximum edge length to the minimum edge length of an element of a triangular mesh
  - For quadrilateral elements
    - AR is defined as the ratio of the maximal diagonal $d$ of the element to the average value $h$ of heights $h_1$ and $h_2$
    - the angle $\alpha$ between adjacent edges of elements can be considered as a mesh quality parameter
A star (sub-mesh) is defined as a set of elements sharing a node/center, whose boundary forms a polygon.

For each star, its AR values equal $a_1, \ldots, a_n$, probabilistic analysis is applied over the fragmentation intervals $(n_j, n_{j+1}]$ (where $n_0 = 1, \ldots, n_k = a_{\text{max}} = \max$) to calculate the histogram values in the current star and calculate new values $a_1, \ldots, a_n$.
Deterministic Pseudo-random Numbers

The number $F_i$ assigned to the $i^{th}$ element of the mesh with the mesh quality parameter value $a_i$ is considered as a quasi-random number and is used to calculate a corresponding $F_{j(i)}$ distribution number:

$$a_i \mapsto F_i = F_{j(i)} = \frac{\{i : a_i \leq n_{j+1}\}}{n+1}$$
Triangular and Quadrilateral Improvement Algorithm

- **Formula Interpretation**
  - For each number $F_i$ one can produce a new value $a_i^*$ of a mesh quality parameter by the Equation
    \[
    a_i^* = \sqrt{-2D \ln(1 - F_i)} + \mu
    \]
    where $\mu$ is an ideal value for the given mesh quality parameter and $D$ is the deviation (from the average value $M$ of AR or angle)
  - The ideal AR $\mu = 1$ value corresponds to an equilateral triangle element
  - The ideal AR $\mu = 2$ value corresponds to a square element
Triangular and Quadrilateral Improvement Algorithm

- **Construction of a Potential Form for Triangle Elements**
  - Operation on a star:
    - the potential geometrical form for each fixed boundary edge AB in the star is generated according to $a_i^*$ value
    - maximization of the minimum angle ($r = AB / a_i^*$, center of the circle with radius $r$ is placed arbitrary in A or B)
Triangular and Quadrilateral Improvement Algorithm

- **Construction of a Potential Form for Quadrilateral Elements**
  - Consider a quadrilateral element of the star with a fixed boundary (two edges of the quadrilateral)
  - Potential forms of elements are defined for each fixed pair of element edges and supported by the diagonals of the star elements, which form a triangular pyramid inside the star
  - Then the coordinates of all potential centres of the star are averaged and the new point placement is determined
Triangular and Quadrilateral Improvement Algorithm

- **Construction of a Potential Form for Quadrilateral Elements**
  - AR is a mesh quality parameter. The height $h$ can be found with the modeled value and the fixed diagonal $a_i$ of a quadrilateral element of the mesh.
  - The height determines the coordinates of the new potential vertex $P$ of the element.
  - When the angle $\alpha$ is the mesh quality parameter, the situation is similar. The only difference is in the geometrical construction, which is made with the fixed diagonal and modeled value of $\alpha$. 
Triangular and Quadrilateral Improvement Algorithm

- **Local Mesh Improvement**
  - The mesh quality parameter is calculated in 2D space, where the 3D coordinates of each vertex are projected onto the corresponding local tangent plane.
  - One element $\sigma$ with the mesh quality parameter $p$ value is projected onto an element $\sigma_\pi$ with the corresponding value $p_\pi$.
  - It can be shown that the projection of 3D coordinates of vertices onto a local tangent plane is allowed. With some calculation one can obtain an increasing function $p_\pi = h(p)$, which is required.

- Limitation of the algorithm
  - The technique requires that a neighborhood of a point be a closed, oriented manifold embedded in 3-space and has the property that, around every one of its points, there exists a neighborhood that is homeomorphic to a plane.
Mesh quality improvement: space-mapping interpolation approach

- The goal and main contributions:
  - Applying space-mapping interpolation approach based on the use of radial basis functions (RBFs) for the mesh (with a big number of the poorest quality elements) improvement which can provide a “deformation surface” satisfying a minimum bending properties to avoid the drawbacks of existing methods: inverted and bad-shaped elements.
  - Reducing the flatness distortion (warping) of hexahedral mesh element faces.
Mesh quality improvement (Cont.)

- Illustration of the improvement problems  Triangulated surface

(a) Delaunay triangulation of elevation contour data; the fragment of the model “Mount Bandai”, Japan.
(b) After proposed method;
(c) After the quasi-statistical approach QS;
(d) The mesh fragment by Laplacian smoothing.
(f) Illustration of the inverted element.

<table>
<thead>
<tr>
<th>Case</th>
<th>Average AR/D</th>
<th>Num. iterations</th>
<th>% inverted el.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original, 43663 el.</td>
<td>7.03/ 24.19</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Proposed method</td>
<td>4.8/ 11.82</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Laplacian</td>
<td>4.3/ 10.1</td>
<td>70</td>
<td>5</td>
</tr>
<tr>
<td>Statistics QS</td>
<td>5.56</td>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>
Mesh quality improvement (Cont.)

Laplacian smoothing – mesh is considered as a spring system.

\[ x_i = \frac{1}{N} \sum_{j=0}^{N} x_j \]

New node location (averaging neighbor nodes)- spring system is in equilibrium

Centroids of potential forms

New node location by spline

Space-mapping based interpolation technique- minimum bending spline

New node location by averaging potential centers

Statistical approach QS- the new potential centers according to predicted potential forms
Mesh quality improvement (Cont.)

- Method (SMA) is based on local transformation and requires an improvement of mesh quality parameters as an aspect ratio \(\text{AR} = \frac{l_{\text{max}}}{l_{\text{min}}}, \ M\)-the mean of AR), angle, etc.

• Space-mapping in \(\mathbb{R}^n\) defines a relationship between each pair of the points in the original and deformed objects.

✓ Applying space-mapping technique based on RBFs

• Let an \(n\)-dimensional region \(\Omega \subset \mathbb{R}^n\) of an arbitrary configuration be given, and let \(\Omega\) contains a set of arbitrary control points

\[ \{q_i = (q_{1i}, q_{2i}, \ldots, q_{ni}) : i = 1,2, \ldots, N\} \quad \text{for a non-deformed object} \]

\[ \{d_i = (d_{1i}, d_{2i}, \ldots, d_{ni}) : i = 1,2, \ldots, N\} \quad \text{for the deformed object} \]

\(N = k + p\), where

\(k\) is the number of centroids of the initial elements of a star,

\(p\) is the number of the boundary nodes of the star. It is assumed that the points \(q_i\) and \(d_i\) are distinct and given.

• The goal of the construction of the deformed object is to find a smooth mapping function that approximately describes the spatial transformation applied to the centre of a star \(o\).
Mesh quality improvement (Cont.)

The basic steps of the (Space Mapping Algorithm) SMA

1. Construct the potential/deformed forms (the ideal shape) of the elements of a star (a set of elements sharing a node/center, whose boundary forms a polygonal shell). The goal of the construction of the deformed object is to find a smooth mapping function that approximately describes the spatial transformation applied to the centre of a star.

   In 3D volume mesh, the problem of constructing deformed forms is formulated in terms to design new vertex coordinates of a tetrahedron. We consider the problem as the optimization problem where the AR is used as the objective function, so it attains the maximal value for an equilateral tetrahedron.

2. Calculate the coordinates of centroids of the deformed and initial forms.

3. Calculate coordinates of the new center of the star by using space-mapping technique and move old one to the new location. In triangular mesh 3D coordinates of each vertex are projected onto the tangent plane and the local surface properties are estimated.

The starting points $q_i$ are centroids of the initial forms and the destination points $d_j$ are centroids of the potential forms. $o$ is the center of the star.
Mesh quality improvement (Cont.)

- The inverse mapping function for transformation of the centre of a star can be given in the form:

\[ q_i = f(d_i) + d_i, \]

where \( f(d_i) \) are splines interpolating displacements of initial points \( q_i \).

- The spline \( f(P) \) having values \( h_i \) at \( N \) points \( P_i \) is the function:

\[ f(P) = \sum_{j=1}^{N} \lambda_j \phi(||P - P_j||) + p(P), \quad (1) \]

where \( p = \nu_0 + \nu_1 x + \nu_2 y + \nu_3 z \) is a degree-one polynomial (coefficients \( \nu \) correspond to the affine part of the transformation), \( N \) is the number of data points, \( \lambda \)'s are the interpolation coefficients to be determined and \( \phi \) is the radial basis function.

(2D case: \( \phi(r) = r^2 \log(r) \); 3D case: \( \phi(r) = r - \) Euclidean distance between two points).

To solve for the weights \( \lambda_j, \nu_0, \nu_1, \nu_2, \nu_3 \) we have to satisfy the constraints \( h_i \) by substituting the right part of Eq. (1)

\[ h_i = \sum_{j=1}^{N} \lambda_j \phi(||P_i - P_j||) + p(P_i). \]
Mesh quality improvement (Cont.)

• \( \lambda \) and \( \nu \) are the coefficients that satisfy the linear system \( Tx = b \).

\[
T = \begin{bmatrix}
A & B^T \\
B & D
\end{bmatrix} \quad x = [\lambda_1, \lambda_2, ..., \lambda_N, \nu_1, ..., \nu_3]^T, \quad b = [h_1, h_2, ..., h_N, 0, 0, ..., 0]^T.
\]

• The matrix \( T \), which consists of three blocks, square sub-matrices \( A = [\phi(|P_i - P_j|)] \) and \( D = 0 \) of size \( N \times N \) and \( 4 \times 4 \) respectively, and \( B = [p(P_i)] \), which is not necessarily square and has the size \( N \times 4 \).

• Once we have the Gauss LTL (or Householder QR) decomposition of \( T \), we solve with three right-hand sides to obtain \( h \)-values for \( x, y, z \)-directional transformations applied to the centre of a star.

• To prevent a rigid motion of a star, boundary conditions as zero displacements of nodes adjustment to the centre of a star are used.
Mesh quality improvement (Cont.)

• **Triangulated surface**

![Image of mesh elements](image)

The "Fandisk" model. The number of the mesh elements $N$: 12946. (a) The initial model and a mesh fragment; $M = 1.45$, $D=0.018$ (b) The model and the mesh fragment after the SMA (5 iterations), $M = 1.39$, $D=0.02$. The difference in the volume enclosed by the surface is 0.0005%. Time 110 sec.

The aspect ration (AR) is a ratio of the maximal edge length to the minimal length of the triangle. $M$ is a average value of the AR.

### Experimental results

<table>
<thead>
<tr>
<th>Case</th>
<th>Number el.</th>
<th>Average AR</th>
<th>Min angle/deg.</th>
<th>Max angle/deg.</th>
<th>Iterations</th>
<th>Time /sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original (c)</td>
<td>20080</td>
<td>1.796</td>
<td>2.34</td>
<td>162</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SMA (d)</td>
<td>20080</td>
<td>1.44</td>
<td>5.4</td>
<td>146</td>
<td>5</td>
<td>170</td>
</tr>
<tr>
<td>Laplacian (e)</td>
<td>20080</td>
<td>1.46</td>
<td>15</td>
<td>141</td>
<td>120</td>
<td>15</td>
</tr>
</tbody>
</table>
Mesh quality improvement (Cont.)

- **Triangulated surface**

  A "Badmesh" planar model and the AR distribution histograms ((x) axis – the index of the intervals, where the AR values belong to the range \( X = [1, \text{max}] \); (y) axis – the number of the mesh elements). (a) Original mesh, \( M = 2.32 \); (b) The mesh after the SMA; (c) The mesh after Laplacian smoothing.

<table>
<thead>
<tr>
<th>Case</th>
<th>Average AR</th>
<th>Num. iterations/Time</th>
<th>Num. inverted el.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original, 10997 el.</td>
<td>2.32</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SMA</td>
<td>1.51</td>
<td>20/43 sec.</td>
<td>no</td>
</tr>
<tr>
<td>Laplacian</td>
<td>1.71</td>
<td>100/5</td>
<td>49</td>
</tr>
<tr>
<td>Statistics QS</td>
<td>1.56</td>
<td>5/10</td>
<td>5</td>
</tr>
</tbody>
</table>
Mesh quality improvement (Cont.)

- **Tetrahedral mesh**

  - **Experimental results**

  (a) The “Maillage” model. The number of cells: 1097. AR_{\text{min\_original}}=0.0001; AR_{\text{min\_improv}}=0.5683; M_{\text{original}}=0.8; M_{\text{improv}}=0.89. Cut-away view. 15 iteration steps.

  (b) A tetrahedral element of the model.

  The AR is defined as a ratio of the inscribed sphere radius to a circumsphere radius of a tetrahedron.
Mesh quality improvement (Cont.)

- **Tetrahedral mesh**

  *Experimental results*

  Mechanical model 32131 el., Calculation time 23 sec., 1 iteration step. Distortion ratios (MSC.PATRAN)

<table>
<thead>
<tr>
<th>Mesh quality parameters</th>
<th>Number of elements (original mesh)</th>
<th>Number of elements (SMA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AR &gt;10</td>
<td>906</td>
<td>488</td>
</tr>
<tr>
<td>AR&gt; 20</td>
<td>375</td>
<td>169</td>
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<tr>
<td>AR&gt; 50</td>
<td>78</td>
<td>31</td>
</tr>
<tr>
<td>Edge angle &gt;90(^0)</td>
<td>801</td>
<td>448</td>
</tr>
<tr>
<td>Edge skew&gt;75(^0)</td>
<td>2138</td>
<td>1391</td>
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<tr>
<td>Collapse</td>
<td>91</td>
<td>26</td>
</tr>
</tbody>
</table>
Mesh quality improvement (Cont.)

- **Tetrahedral mesh**

> Mechanical model. 205554 el., Calculation time 700 sec.

Distortion ratios (MSC.PATRAN ), M=0.873(after the SMA)

<table>
<thead>
<tr>
<th>Mesh quality parameters</th>
<th>Number of elements (original mesh)</th>
<th>Number of elements (SMA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AR&gt;10</td>
<td>203</td>
<td>161</td>
</tr>
<tr>
<td>AR&gt;20</td>
<td>174</td>
<td>63</td>
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<td>AR&gt;50</td>
<td>32</td>
<td>17</td>
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<tr>
<td>Edge angle&gt;90°</td>
<td>202</td>
<td>158</td>
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<tr>
<td>Edge angle&gt;75°</td>
<td>385</td>
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<tr>
<td>Edge skew&gt;75°</td>
<td>204</td>
<td>196</td>
</tr>
<tr>
<td>Collapse</td>
<td>120</td>
<td>53</td>
</tr>
</tbody>
</table>

![The AR histogram of distribution](image)

Experimental results

<table>
<thead>
<tr>
<th>Model</th>
<th>Eigenvalues analysis calculation time (sec.)</th>
<th>Number of elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>“Bed” initial</td>
<td>650</td>
<td>205564 (modified)</td>
</tr>
<tr>
<td>“Bed” improved after 15 iterations</td>
<td>635</td>
<td>205564</td>
</tr>
</tbody>
</table>
Mesh quality improvement (Cont.)

- **Tetrahedral mesh**

  The "Detail" model. 25335 el.; Calculation time 90 sec., 1 iteration. Combining the surface and volume mesh improvement.

  (a) The initial mesh and zoom fragment. $M=0.554$; (b) The mesh after 1 improvement step and the same zoom fragment. $M=0.572$. $M$ is increased by 3.5%.
Mesh quality improvement Cont.

Combine technique CSMA and Laplacian-based technique AV for the “Mechanical” model (MM)

<table>
<thead>
<tr>
<th></th>
<th>MM</th>
<th>MM.CSMA</th>
<th>MM.AV</th>
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</thead>
<tbody>
<tr>
<td>ARmin</td>
<td>7.3E-8</td>
<td>2.2E-6</td>
<td>2.4E-7</td>
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<tr>
<td>ARmax</td>
<td>0.997</td>
<td>0.996</td>
<td>0.997</td>
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<tr>
<td>M</td>
<td>0.655</td>
<td>0.686</td>
<td>0.685</td>
</tr>
<tr>
<td>#[1]</td>
<td>1083</td>
<td>542</td>
<td>545</td>
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<td>#[2]</td>
<td>916</td>
<td>592</td>
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<td>1198</td>
<td>961</td>
<td>878</td>
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<td>#[4]</td>
<td>1727</td>
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<td>2338</td>
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<td>#[10]</td>
<td>3599</td>
<td>4247</td>
<td>4188</td>
</tr>
</tbody>
</table>
Mesh quality improvement (Cont.)

- **Hexahedral mesh**

✓ In the case of all-hexahedral mesh, a star is the set of quadrilateral faces of a hexahedron sharing a node in 3D space. An ideal case: 8 hexahedrals with 12 quadrilaterals.
✓ The angle between adjacent edges of the quadrilateral elements is considered as well as a mesh quality parameter.
✓ Starting and deformed centroids \( q_i \), \( d_i \) are calculated for initial and potential forms of the triangle elements.
✓ Edges of hexahedral element and faces must be orthogonal as far as possible.
✓ Calculating the new position of the center of the star according to space-mapping technique. Control points: centroids of the initial star’s elements; boundary nodes of the star.

![A quadrilateral star](image1.png) ![The operational planes](image2.png) ![Potential form](image3.png) ![The star with the potential forms](image4.png)
Mesh quality improvement (Cont).

- **Hexahedral mesh**
  - **Problem statement of the flatness distortion:** Mesh generators and improvement techniques can generate quadrilateral faces of the hexahedral element which are not flat (warp).
  - **Reducing the flatness distortion of quadrilateral faces:**
    Relocation the center of the star $O$ to a new position $O'$ by applying a space mapping approach to the center of a star in accordance to the distances between the diagonals of quadrilaterals/the points $d_i$ and $q_i$. 

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**Diagram:**
- $D_{13}$
- $D_{42}$
- $O$, $O'$
- $d$, $q$
Mesh quality improvement (Cont.)

- **Hexahedral mesh**

**Experimental results**

The “Cube” model. 450 quad. faces; Calculation time 0.52 sec.

(a) The initial mesh of the model. (b) The mesh after statistical improvement QS, 4 iterations. (c) The mesh after the SMA, 4 iterations.

<table>
<thead>
<tr>
<th>Case</th>
<th>Min angle</th>
<th>Max Angle</th>
<th>Average angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original</td>
<td>14°</td>
<td>180°</td>
<td>1.564 80°</td>
</tr>
<tr>
<td>Statistical QS</td>
<td>50°</td>
<td>118°</td>
<td>1.567 89°</td>
</tr>
<tr>
<td>SMA</td>
<td>55°</td>
<td>110°</td>
<td>1.57 90°</td>
</tr>
</tbody>
</table>
Mesh quality improvement (Cont.)

- **Hexahedral mesh**

  Experimental results
  The “Mechanical model” 18030 quad. faces. Calculation time 99 sec.

(a) The model; (b) The initial mesh: Average angle = 1.56 (80 deg.), D = 0.12; (c) The mesh after the SMA: Average angle = 1.57 (90 deg.), D = 0.09, 1 iteration. (d) The mesh after quasi-statistical improvement (QS): Average angle = 1.566 (<90 deg.), D = 0.11, (1 iteration).
Mesh quality improvement (Cont.)

- **Hexahedral mesh**  
  
  **Experimental results**

The “Mechanical model”. Quality elements measures by PATRAN.

<table>
<thead>
<tr>
<th>Mesh quality parameters</th>
<th>Number of hex elements</th>
</tr>
</thead>
<tbody>
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<th>Number of hex elements</th>
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The initial mesh

The improved mesh
Mesh quality improvement (Cont.)

- **Hexahedral mesh**

Experimental results

The “Mechanical model”. Angle distribution histograms.

(a) The initial mesh; (b) The mesh after improvement. (x) axis – the values of the angle; (y) axis – the number of the mesh elements.
Mesh quality improvement (Cont.)

**Conclusions**

- A new approach based on RBFs space-mapping technique for improving the mesh quality is presented.
- This technique can improve the smoothness of the isotropic surface meshes without distorting the initial surface without an appearance of the inverted elements with providing the volume and features preservations.
- Proposed smoothing method is applicable to triangular, quadrilateral, and tri-quad mixed mesh.
- Implementation of this technique for tetrahedral mesh shows the effectiveness of it to improve mesh elements with the initial volume closed to 0.
- In hexahedral meshes suggested method allows to reduce the maximum flatness deviation of the quadrilateral elements in the hexahedral meshes.
- The method works well with highly degraded meshes.
- This method can be combined with traditional smoothing techniques for improving the meshes of the thin-shape and very bad-shape models.
- The geometrical improving approach can be combine with topological methods for receiving better results.
Appendix 1

Aspect Ratio (AR) for a tet element is the ratio of the height $h_i$ of a vertex to the square root of the area of the opposing face $A_i$. 
Edge Angle measures the maximum deviation angle between adjacent faces of a solid element (in an equilateral tetrahedral element this parameter is equal 70.259 deg.)
**Appendix 1** (Cont)

**Edge Skew** measures angular deviation from ideal shape in surface elements. The difference is taken of the angle between edge bisectors and 90 deg.
Collapse is the ratio of the height of a vertex to the square root of the area of the opposing face. The value approaches zero as the volume of the element approaches zero. Collapse is an indicator of near zero volume tetrahedral elements. This value approaches zero as the volume of the element approaches zero.
Appendix 1 (Cont)

**Hexahedral element**: Aspect Ratio is calculated as the ratio of the distance between opposing faces. The Aspect Ratio is determined by taking the maximum distance between any two faces and dividing it by the minimum distance between them.

\[
\text{Aspect Ratio} = \frac{\max (h_1, h_2, h_3)}{\min (h_1, h_2, h_3)}.
\]
Appendix 1 (Cont)

**Edge Angle.** Edge angle measures the maximum deviation angle between the two faces meeting at an edge subtracted from the ideal angle for that edge of a solid element.

\[
\text{Edge Angle} = \text{Max} (900 - \alpha).
\]

**Face Twist** is the rotation of one face of the solid with respect to its opposite face. A twist angle is computed about all three principal axes of hex elements. To compute the twist angle, each face is treated as if it were a warped quad. Vectors from the center of the projected plane to the middle of two adjacent edges are constructed.
Face Skew. Each face of the hex element is tested for skew as if it were a quad element. The skew measures angular deviation from a rectangular shape in surface element.

Face Warp. Each face of the hex element is tested for warp as if were a quad element. The face warp coefficient test tolerance is the cosine of the angle formed between the normal vectors located at diagonally opposite corner points on each face surface. This value is 1.0 for a plane face.

Face Taper. Taper ratio for the quadrilateral element is defined to be the ratio of the area with the smallest triangle and the total area of the element. The triangles are formed at each corner grid point. The ratio approaches 1.0 the shape approaches a rectangle.
Appendix 2. Elements of general theory of splines

- The following minimum condition \( \int_a^b [u''(x)]^2 dx \) is equal to the minimization of bending energy.
- Given a set of data points, a weighted combination of thin-plate splines centered about each data point gives the interpolation function that passes through the points exactly while minimizing the so-called bending energy.
- For n-dimensional arbitrary area \( \Omega \), which contains a set of points \( P_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \): \( i = 1, 2, \ldots, N \), the development of the theory was to construct the interpolation spline function \( U(P_i) \in W_2^m(\Omega) \), where \( W_2^m \) is the set of all functions whose squares of all derivatives of order \( \leq m \) are integrable over \( \mathbb{R}^n \), so that \( U(P_i) = h_i \), \( i = 1, 2, \ldots, N \), and has minimum energy of all functions that interpolate the values \( h_i \). It leads to the utilization of differential operator \( D \) and using more general minimum condition

\[
\Phi(U) = \int \sum_{\alpha=0}^{m!} m!/\alpha! (D^\alpha U)^2 \, d\Omega \rightarrow \text{min}
\]

\( U \) minimizes a bending energy, if the space transformation is seen as an elastic deformation.
Appendix 2. (Cont) Elements of general theory of splines

- Minimization can be done by the use of the Green’s function (RBFs). In the 2D case, it is a so-called thin plate function. $G(x,x^i)$ is the Green function associated with the differential operator $T$. $G(x,x^i)$ is two point function which depends on $x$ and $x^i$. The Green’s function for the operator $T^*T$ where

$$Tu = \sqrt{m!/\alpha!} \ D^\alpha u$$

has the form:

$$G_{m,n}(x,P_i) = \begin{cases} 
\|x - P_i\|^{2m-n} \ln\|x - P_i\|, n - \text{even}, \\
\|x - P_i\|^{2m-n}, n - \text{odd},
\end{cases}$$

where $x = (x_1, x_2, ..., x_n)$ is an arbitrary point of $E^n$, $n$ is dimension of the underlying space.

$$\|x - P_i\| = \sum_{j=1}^{n} [ (x_j - x_i^j)^2 ]^{1/2}.$$
Appendix 2. (Cont) Elements of general theory of splines

- The coefficient $m \geq 2$ defines a norm. In practice, $m = 2$ can be used. For $m = 2$ and $n = 2, 3$ the spline (hereafter we call it volume spline, if $n = 3$) has the following form:

$$U(x) = \sum_{i=1}^{N+k} \lambda_i g_i(x, P_i),$$

where $g_i (x, P_i) = G_{m, n}(x, P_i), i = 1, ..., N,$

$g_{N+1}(x, P_i) = 1,$ $g_{N+1+j}(x, P_i) = x_j, j = 1, ..., k - 1,$ $k = (n + m - 1)! / (n! (m - 1)!)$. If $m = n = 2$, then $k = 3$, and it is sufficient to find three points not lying in a straight line.

- The spline coefficients $\lambda_i$ are calculated using the system of $(N + k)$ linear equations: $T \lambda = h$
Appendix 2. (Cont) Elements of general theory of splines

\[
T = \begin{bmatrix}
\lambda_1 & h_1 \\
\lambda_2 & h_2 \\
\vdots & \vdots \\
\lambda_N & h_N \\
\lambda_{N+1} & 0 \\
\vdots & \vdots \\
\lambda_{N+k} & 0 \\
\end{bmatrix}
\]

• The components of the matrix \( T \) are: \( T_{ij} = g_i(P_i, P_j), i \leq N + k, j \leq N, i < j, i > j; \)
  \( T_{ii} = 0, i < N; \)
  \( T_{ij} = g_j(P_i, P_j), i \leq N, N < j \leq N+k; \)
  \( T_{ij} = 0, i > N, j > N. \)

• The only problem here is operating with the dense matrix of size \( N \) with zero diagonal elements. This matrix is symmetric, but not positive definite. The system may be solved, for instance, by the Householder method. After \( \lambda \) calculating the spline \( U(x) \) can be restored and provides \( C_k \)-continuity if \( k < 2m - n \).

Motivation of using the volume spline: it is used for the case of scattered points; minimal energy property; \( C_k \) continuity with \( k<2m-n \).